

ON TILTING OBJECTS FOR THE WEIGHTED PROJECTIVE LINES OF GENUS ONE

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ABSTRACT. We investigate the tilting objects in the stable category of vector bundles on a weighted projective line of genus one and realize the classifications of the endomorphism algebras of the tilting objects in the category of coherent sheaves on the weighted projective line of type $(2, 2, 2, 2; \lambda)$ and in its derived category via cluster tilting theory.

1. INTRODUCTION

Let \mathbb{X} be a weighted projective line of genus one over an algebraically closed field k . A famous theorem of Lenzing and Meltzer [17] points that \mathbb{X} is of weight type $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$ or $(2, 2, 2, 2; \lambda)$, where λ is a parameter in the projective line $\mathbb{P}^1(k)$ different from $0, 1, \infty$. Kussin, Lenzing and Meltzer [16] proved that the category $\text{vect } \mathbb{X}$ of vector bundles on \mathbb{X} , under the exact structure called distinguished exact, is a Frobenius category with the system \mathcal{L} of all line bundles as the system of all indecomposable projective-injectives. A general result of Happel [11] asserts that the attached stable category

$$\underline{\text{vect}} \mathbb{X} = \text{vect } \mathbb{X} / [\mathcal{L}]$$

is triangulated. Here, the stable category $\text{vect } \mathbb{X} / [\mathcal{L}]$ is the category whose objects are the objects of $\text{vect } \mathbb{X}$ and the set of morphisms is given by all morphisms factoring through a finite direct sum of line bundles.

Much of the work on this triangulated category $\underline{\text{vect}} \mathbb{X}$ has focused on the tilting objects (see for instance [16, 8]). Kussin, Lenzing and Meltzer [16] showed the existence of a tilting object in $\underline{\text{vect}} \mathbb{X}$ for the weighted projective lines of three weights. Chen, Lin and Ruan [8] constructed two tilting objects in $\underline{\text{vect}} \mathbb{X}$ of weight type $(2, 2, 2, 2; \lambda)$. In this way, it seems a little hard to find more tilting objects directly in $\underline{\text{vect}} \mathbb{X}$.

The Fomin-Zelevinsky mutation of quivers plays an important role in the theory of cluster algebras initiated in [9]. Motivated by this theory, a mutation of cluster tilting objects in cluster categories, and more generally Hom-finite triangulated 2-Calabi-Yau categories has been investigated in [6, 12]. This has turned out to give a categorical model for the quiver mutation in certain cases [7, 5]. From the viewpoint of mutation, an advantage of cluster tilting theory over classical tilting theory is that in cluster tilting theory it is always to mutate. Thus the usual procedure of going from a tilting object to another one by exchanging just one indecomposable direct summand gets more regular.

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The main idea of this article is to use cluster tilting theory to investigate the tilting objects in $\underline{\text{vect}}\mathbb{X}$. We describe the relationship between the tilting objects in $\underline{\text{vect}}\mathbb{X}$ and the cluster tilting objects in the orbit category of $\underline{\text{vect}}\mathbb{X}$ under the action of the unique auto-equivalence G satisfying

$$D\underline{\text{Hom}}(X, Y[1]) = \underline{\text{Hom}}(GY, X[1])$$

where D denotes the usual duality functor $\text{Hom}_k(-, k)$ and $[1]$ the suspension functor of $\underline{\text{vect}}\mathbb{X}$. We prove that the (cluster) tilting objects in $\underline{\text{vect}}\mathbb{X}/G^{\mathbb{Z}}$ are one to one corresponding to the tilting objects in $\underline{\text{vect}}\mathbb{X}$ with the slopes of the indecomposable direct summands belonging to some given interval. Following this, for the weight type $(2, 2, 2, 2; \lambda)$, we realize, independently by Meltzer [18], a complete classification of the endomorphism algebras of tilting objects in $\underline{\text{vect}}\mathbb{X}$. Furthermore, we classify all the endomorphism algebras of tilting sheaves in the category $\text{coh}\mathbb{X}$ of weight type $(2, 2, 2, 2; \lambda)$ at the end of the paper.

The paper is organized as follows: In section 2, we recall some basic results on the category of vector bundles on a weighted projective line. In section 3, we study the relationship between the tilting objects in the stable category of vector bundles on a weighted projective line of genus one and the cluster tilting objects in the associated cluster category. We focus on the weighted projective line of type $(2, 2, 2, 2; \lambda)$ in section 4. We show that each tilting object in the stable category pushes to a cluster tilting object in its cluster category, and describe all the tilting objects corresponding to a given cluster tilting object in the cluster category. Moreover, we realize the classifications of all the endomorphism algebras of the tilting objects in the category of coherent sheaves and in its derived category.

Throughout the paper, we view the isomorphism as equality.

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2. STABLE CATEGORY OF VECTOR BUNDLES

In this section, we present some materials concerning the category of vector bundles over a weighted projective line.

2.1. Coherent sheaves on the weighted projective line. A *weighted projective line* \mathbb{X} is specified by giving a collection $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of distinct points in the projective line $\mathbb{P}^1(k)$, and a *weight sequence* $p = (p_1, p_2, \dots, p_t)$, that is, a sequence of positive integers. Let \mathbb{L} be the rank one abelian group with generators $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_t$ and the relations

$$p_1\vec{x}_1 = p_2\vec{x}_2 = \dots = p_t\vec{x}_t =: \vec{c}.$$

Then each $\vec{x} \in \mathbb{L}$ can be uniquely written in normal form

$$\vec{x} = \sum_{i=1}^t l_i \vec{x}_i + l \vec{c}, \quad \text{where } 0 \leq l_i < p_i \text{ and } l \in \mathbb{Z}.$$

In addition, if \vec{x} is in normal form as above, one can define

$$\vec{x} \geq 0 \quad \text{if and only if } l \geq 0.$$

Then \mathbb{L} becomes a partial order group, and each $\vec{x} \in \mathbb{L}$ satisfies exactly one of the two possibilities:

$$\vec{x} \geq 0 \quad \text{or} \quad \vec{x} \leq \vec{\omega} + \vec{c},$$

where $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^t \vec{x}_i$. The element \vec{c} is called the *canonical element* of \mathbb{L} and $\vec{\omega}$ is called the *dualizing element*.

Denote by S the commutative algebra

$$S = k[X_1, X_2, \dots, X_t]/I = k[x_1, x_2, \dots, x_t],$$

where $I = (f_3, \dots, f_t)$ is the ideal generated by $f_i = X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}$, $i = 3, \dots, t$. Then S is \mathbb{L} -graded by setting $\deg(x_i) = \vec{x}_i$ ($i = 1, 2, \dots, t$). Hence S carries a decomposition

$$S = \bigoplus_{\vec{x} \in \mathbb{L}} S_{\vec{x}}$$

into k -subspaces.

The category, $\text{coh } \mathbb{X}$, of coherent sheaves on \mathbb{X} , can be defined as the quotient of the category of finitely generated \mathbb{L} -graded S -modules by the Serre subcategory of finite length modules

$$\text{coh } \mathbb{X} = \text{mod}^{\mathbb{L}}(S)/\text{mod}_0^{\mathbb{L}}(S).$$

Geigle and Lenzen [10] showed that $\text{coh } \mathbb{X}$ is a hereditary abelian category; the free module S gives the structure sheaf \mathcal{O} , and shifting the grading gives twists $E(\vec{x})$ for any sheaf E and $\vec{x} \in \mathbb{L}$. Each line bundle has the form $\mathcal{O}(\vec{x})$ for a uniquely determined $\vec{x} \in \mathbb{L}$, and there are natural isomorphisms

$$\text{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}.$$

Also, every coherent sheaf is the direct sum of a vector bundle, that is, a sheaf with a filtration by line bundles, and a finite length sheaf. By $\text{vect } \mathbb{X}$ we denote the full subcategory of the category $\text{coh } \mathbb{X}$ formed by all vector bundles. There are no non-zero morphisms from any finite length sheaf to $\text{vect } \mathbb{X}$. Moreover, the category $\text{coh } \mathbb{X}$ satisfies Serre duality, i.e. bifunctorial isomorphism

$$D \text{Ext}^1(X, Y) = \text{Hom}(Y, X(\vec{\omega})).$$

It implies the existence of almost split sequences for $\text{coh } \mathbb{X}$ with the Auslander-Reiten translation τ given by the shift with $\vec{\omega}$.

The Grothendieck group $K_0(\mathbb{X})$ of $\text{coh } \mathbb{X}$ was computed by Geigle and Lenzen [10], and it was proved to be the vector space with basis indexed by elements $\mathcal{O}(\vec{x})$ with $0 \leq \vec{x} \leq \vec{c}$, where we still write $X \in K_0(\mathbb{X})$ for the class of an object $X \in \text{coh } \mathbb{X}$. The *determinant* is the unique group homomorphism $\det : K_0(\mathbb{X}) \rightarrow \mathbb{L}$ such that

$$\det(\mathcal{O}(\vec{x})) = \vec{x}.$$

Let $\bar{p} = \text{l.c.m.}(p_1, p_2, \dots, p_t)$ and $\delta : \mathbb{L} \rightarrow \mathbb{Z}$ be the homomorphism defined by

$$\delta(\vec{x}_i) = \frac{\bar{p}}{p_i}.$$

There are two important \mathbb{Z} -linear functions, rank rk and degree \deg , on $K_0(\mathbb{X})$. The degree function is the composition of δ and \det , that is, \deg is determined by

$$\deg(\mathcal{O}(\vec{x})) = \delta(\vec{x}).$$

The rank function $\text{rk} : K_0(\mathbb{X}) \rightarrow \mathbb{Z}$ is characterized by

$$\text{rk}(\mathcal{O}(\vec{x})) = 1.$$

For each non-zero object $X \in \text{coh } \mathbb{X}$, define the *slope* of X as

$$\mu X = \frac{\deg(X)}{\text{rk}(X)}.$$

In more detail, the rank is strictly positive on non-zero vector bundles, then the slope belongs to \mathbb{Q} ; and the rank vanishes on finite length sheaves, then the slope is ∞ .

From now on, we shall always assume that

- \mathbb{X} is a weighted projective line of genus one, or equivalently,
- \mathbb{X} is of weight type $(2,3,6)$, $(2,4,4)$, $(3,3,3)$ or $(2,2,2,2;\lambda)$, where λ is a parameter in the projective line $\mathbb{P}^1(k)$ different from $0, 1, \infty$.

Then by [10], for any two indecomposable objects X, Y in $\text{coh } \mathbb{X}$,

$$(2.1) \quad \text{Hom}(X, Y) \neq 0 \text{ implies } \mu X \leq \mu Y.$$

Define the Euler form of X, Y by

$$\langle X, Y \rangle = \dim_k \text{Hom}(X, Y) - \dim_k \text{Ext}^1(X, Y).$$

We have the following.

Theorem 2.1 (Riemann-Roch Formula, [17]). *For each $X, Y \in \text{coh } \mathbb{X}$, we have*

$$\sum_{i=0}^{\bar{p}-1} \langle \tau^i X, Y \rangle = \text{rk}(X) \deg(Y) - \deg(X) \text{rk}(Y).$$

In particular, if $X, Y \in \text{vect } \mathbb{X}$, then

$$(2.2) \quad \sum_{i=0}^{\bar{p}-1} \langle \tau^i X, Y \rangle = \text{rk}(X) \text{rk}(Y) (\mu Y - \mu X).$$

The following lemma was given in [16].

Lemma 2.2. [16, Lemma 5.3] *There is a bijective, monotonous map $\alpha : \mathbb{Q} \rightarrow \mathbb{Q}$ with $\alpha(q) > q$ for all $q \in \mathbb{Q}$ and such that $\mu(X[1]) = \alpha(\mu(X))$ for each indecomposable $X \in \text{vect } \mathbb{X}$.*

Moreover we have the following tubular factorization property.

Lemma 2.3. [16, Theorem A.6] *Let X and Y be indecomposable in $\text{vect } \mathbb{X}$ with slopes $\mu(X) = q$ and $\mu(Y) = q'$. If $q' > \alpha(q)$ then every morphism $X \rightarrow Y$ factors through a direct sum of line bundles.*

2.2. Stable category of vector bundles. Kussin, Lenzing and Meltzer [16] defined a sequence

$$0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$$

in $\text{vect } \mathbb{X}$ to be *distinguished exact* if for each line bundle L the induced sequence

$$0 \rightarrow \text{Hom}(L, X') \rightarrow \text{Hom}(L, X) \rightarrow \text{Hom}(L, X'') \rightarrow 0$$

is exact, and proved the distinguished exact sequences defined an exact structure on the category $\text{vect } \mathbb{X}$ which is Frobenius such that the system of all line bundles is the system of all indecomposable projective-injectives. Therefore, the related stable category

$$\underline{\text{vect}} \mathbb{X} = \text{vect } \mathbb{X} / [\mathcal{L}],$$

is a triangulated category. Denoted by \mathcal{D} the stable category $\underline{\text{vect}} \mathbb{X}$ and by [1] its suspension functor. Moreover, the triangulated category \mathcal{D} is proved to be Hom-finite and homologically finite with Serre duality induced from the Serre duality of $\text{coh } \mathbb{X}$. That is,

Lemma 2.4 ([16]). (1) *\mathcal{D} is Hom-finite: For any two objects X and Y in \mathcal{D} , the space of morphisms $\mathcal{D}(X, Y)$ is finite-dimensional. It also ensures that \mathcal{D} is a Krull-Schmidt category.*

- (2) *\mathcal{D} is homologically finite: For any two objects X and Y in \mathcal{D} , $\mathcal{D}(X, Y[n]) = 0$ for $|n| \gg 0$.*

(3) \mathcal{D} admits a Serre functor: For any two objects X and Y in \mathcal{D} ,

$$\mathcal{D}(X, Y[1]) = D\mathcal{D}(Y, X(\vec{\omega})),$$

In particular, \mathcal{D} has Auslander-Reiten triangles, and the shift by $\vec{\omega}$ also serves as the AR-translation for \mathcal{D} .

The following result is also taken from [16].

Lemma 2.5 (Interval category, [16]). *For any $a \in \mathbb{Q}$, the interval category $\mathcal{D}(a, \alpha(a))$, the full subcategory of \mathcal{D} obtained as the additive closure of all the indecomposable objects with slopes in the interval $(a, \alpha(a))$, is an abelian category and equivalent to the category $\text{coh } \mathbb{X}$.*

An object T is called *tilting* in \mathcal{D} if

- T is extension-free, i.e. $\mathcal{D}(T, T[n]) = 0$ for each non-zero integer n .
- T generates the triangulated category \mathcal{D} , i.e., the smallest triangulated subcategory $\langle T \rangle$ containing T is \mathcal{D} .

In our case, the indecomposable direct summands of extension-free object T in \mathcal{D} can be ordered in such a way that they form an exceptional sequence. Then by [3, Theorem 3.2] and [4, Proposition 1.5], \mathcal{D} is generated by $\langle T \rangle$ together with $\langle T \rangle^\perp$. Hence the 2nd axiom above can be replaced by the following statement (see for instance [16]):

- for each object $X \in \mathcal{D}$, there exists some integer n , such that

$$\mathcal{D}(T, X[n]) \neq 0.$$

Lemma 2.6. *Let $T = \oplus T_i$ be an object in \mathcal{D} with indecomposable direct summand $T_i \in \mathcal{D}(a, \alpha(a))$ for some $a \in \mathbb{Q}$. Then T is extension-free in \mathcal{D} if and only if $\mathcal{D}(T, T[1]) = 0$.*

Proof. For any $T_i, T_j \in \mathcal{D}(a, \alpha(a))$, if $n \leq -1$, then by Lemma 2.2,

$$\mu(T_j[n]) \leq a < \mu(T_i).$$

Hence from (2.1), $\text{Hom}_{\text{coh } \mathbb{X}}(T_i, T_j[n]) = 0$, which implies that

$$(2.3) \quad \mathcal{D}(T_i, T_j[n]) = 0.$$

If $n \geq 2$, then

$$\mu(T_j[n]) \geq \mu(T_j[2]) > \alpha(\mu(T_i)).$$

By Lemma 2.3, each morphism $T_i \rightarrow T_j[n]$ factors through a direct sum of line bundles, which implies that

$$(2.4) \quad \mathcal{D}(T_i, T_j[n]) = 0.$$

Combining 2.3 and 2.4, we finish the proof. \square

3. RELATIONSHIP TO CLUSTER TILTING OBJECTS

Cluster categories were defined in [6] in order to use categorical methods to give a conceptual model for the combinatorics of cluster algebras [9]. For this purpose, a tilting theory was developed in the cluster category and indeed there has been a considerable amount of activity and a lot of results in this direction. We refer to the survey [14, 20]. In this section, we investigate the tilting objects in the stable category of vector bundles on a weighted projective line of genus one via cluster tilting theory.

3.1. Cluster categories. Let H be a hereditary algebra. In [6] the cluster category \mathcal{C}_H is defined to be the orbit category of the bounded derived category of finite-dimensional right H -modules under the action of the auto-equivalence $G = \tau^{-1}[1]$. Following this, Barot, Kussin and Lenzing [2] did manage to study the cluster category of a canonical algebra A in terms of the hereditary category of coherent sheaves over the corresponding weighted projective line \mathbb{X} . Namely they investigated the orbit category $D^b(\text{coh } \mathbb{X})/G^{\mathbb{Z}}$.

According to [19], the bounded derived category of coherent sheaves over the weighted projective line $D^b(\text{coh } \mathbb{X})$ is triangular equivalent to the stable category of vector bundles \mathcal{D} . Thus parallel to [2], we define the cluster category \mathcal{C} to be the orbit category of the stable category \mathcal{D} under the action of the unique auto-equivalence G satisfying

$$D\mathcal{D}(X, Y[1]) = \mathcal{D}(GY, X[1]).$$

According to section 2.2, the functor G is $\tau^{-1}[1]$. The cluster category \mathcal{C} has the same objects as \mathcal{D} , and for any objects X, Y , morphism spaces are given by

$$\mathcal{C}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}(X, G^n Y)$$

with the obvious composition. This orbit category is triangulated and Calabi-Yau of CY-dimension 2, and the canonical functor $\pi : \mathcal{D} \rightarrow \mathcal{C}$ is a triangulated functor, cf. [13]. We still denote by [1] its suspension functor.

3.2. Relationship to cluster tilting objects. Recall from [15] that, an object T in \mathcal{C} is called *rigid* if $\mathcal{C}(T, T[1]) = 0$; and a rigid object T in \mathcal{C} is called a *cluster tilting* object if $\mathcal{C}(T, X[1]) = 0$ implies $X \in \text{add}(T)$.

Throughout this subsection, $T = \bigoplus T_i$ is an object in \mathcal{D} with each indecomposable direct summand $T_i \in \mathcal{D}(a, \alpha(a))$ for some $a \in \mathbb{Q}$, and we use the same notation for T and its image in \mathcal{C} under the canonical functor $\pi : \mathcal{D} \rightarrow \mathcal{C}$.

Lemma 3.1. *The object T is extension-free in \mathcal{D} if and only if T is rigid in \mathcal{C} .*

Proof. By definition and Lemma 2.3, we have

$$\mathcal{C}(T, T[1]) = \bigoplus_{i \in \mathbb{Z}} \mathcal{D}(T, G^i T[1]) = \mathcal{D}(T, T(\vec{\omega})) \oplus \mathcal{D}(T, T[1]).$$

According to Lemma 2.4,

$$\mathcal{D}(T, T[1]) = D\mathcal{D}(T, T(\vec{\omega})).$$

Hence

$$\mathcal{D}(T, T[1]) = 0 \text{ if and only if } \mathcal{C}(T, T[1]) = 0.$$

By Lemma 2.6, T is extension-free in \mathcal{D} if and only if T is rigid in \mathcal{C} . \square

The following lemma is crucial for us to describe the relationship between the tilting objects in \mathcal{D} and the cluster tilting objects in \mathcal{C} .

Lemma 3.2. *If T is a cluster tilting object in \mathcal{C} , then $T(\vec{\omega}) \oplus T(-\vec{\omega}) \in \langle T \rangle$.*

Proof. It suffices to show that for each indecomposable direct summand T_i of T ,

$$T_i(\vec{\omega}) \oplus T_i(-\vec{\omega}) \in \langle T \rangle.$$

By Lemma 2.4, $\mathcal{D}(T_i, T_i(\vec{\omega})[1]) = D\mathcal{D}(T_i, T_i) \neq 0$, which implies $T_i(\vec{\omega}) \in \langle T \rangle$.

In the following we show that $T_i(-\vec{\omega}) \in \langle T \rangle$. If $\mathcal{D}(T, T_i(-\vec{\omega})) \neq 0$, we have done. If else, there are two cases to consider:

Case 1: $T_i(-2\vec{\omega}) = T' \in \text{add}(T)$ and therefore $T_i(-\vec{\omega}) = T'(\vec{\omega}) \in \langle T \rangle$.

Case 2: $T_i(-2\vec{\omega}) \notin \text{add}(T)$. It follows that

$$0 \neq \mathcal{C}(T, T_i(-2\vec{\omega})[1]) = \mathcal{D}(T, T_i(-2\vec{\omega})[1]) \oplus \mathcal{D}(T, T_i(-\vec{\omega})).$$

Then the assumption $\mathcal{D}(T, T_i(-\vec{\omega})) = 0$ implies that

$$\mathcal{D}(T, T_i(-2\vec{\omega})[1]) \neq 0.$$

Let $\theta : \bar{T}(\vec{\omega}) \rightarrow T_i(-\vec{\omega})[1]$ be the minimal right $\text{add}(T(\vec{\omega}))$ -approximation of $T_i(-\vec{\omega})[1]$ and

$$(3.1) \quad \bar{T}(\vec{\omega}) \xrightarrow{\theta} T_i(-\vec{\omega})[1] \rightarrow Y \rightarrow \bar{T}(\vec{\omega})[1]$$

be the induced triangle. Thus $T_i(-\vec{\omega}) \in \langle T \rangle$ if $Y \in \langle T \rangle$.

We claim that Y does be in $\langle T \rangle$. In fact, let Y_j be any indecomposable direct summand of Y . If $\mathcal{D}(T, Y_j[-1]) \neq 0$, then $Y_j \in \langle T \rangle$. Now we suppose that

$$(3.2) \quad \mathcal{D}(T, Y_j[-1]) = 0.$$

Applying $\mathcal{D}(T(\vec{\omega}), -)$ to the triangle (3.1), we obtain the following exact sequence

$$\mathcal{D}(T(\vec{\omega}), T_i'(\vec{\omega})) \xrightarrow{\mathcal{D}(T(\vec{\omega}), \theta)} \mathcal{D}(T(\vec{\omega}), T_i(-\vec{\omega})[1]) \rightarrow \mathcal{D}(T(\vec{\omega}), Y) \rightarrow 0.$$

Note that θ is the minimal right $\text{add}(T(\vec{\omega}))$ -approximation, so $\mathcal{D}(T(\vec{\omega}), \theta)$ is surjective. It follows that

$$(3.3) \quad \mathcal{D}(T(\vec{\omega}), Y) = 0.$$

Again by applying $\mathcal{D}(G^n T(\vec{\omega}), -)$ to the induced triangle (3.1), we have

$$\mathcal{D}(G^n T(\vec{\omega}), Y) = 0 \text{ for any } n \neq 0, 1.$$

Combining with (3.2) and (3.3), we get

$$\mathcal{C}(T, Y_j(-\vec{\omega})) = \bigoplus_{n \in \mathbb{Z}} \mathcal{D}(G^n T, Y_j(-\vec{\omega})) = 0.$$

It follows that $Y_j(-\vec{\omega})[-1] \in \text{add}(T)$ since T is a cluster tilting object in \mathcal{C} . Hence there exists some $T'' \in \text{add}(T)$ such that $Y_j = T''(\vec{\omega})[1] \in \langle T \rangle$. So we have $Y \in \langle T \rangle$. This finishes the proof. \square

The main result of this section is the following.

Theorem 3.3. *The object T is a tilting object in \mathcal{D} if and only if T is a cluster tilting object in \mathcal{C} .*

Proof. Firstly, we show the “if” part. By Lemma 3.1 and Lemma 2.2, it is sufficient to show $X \in \langle T \rangle$ for each indecomposable object $X \in \mathcal{D}(a, \alpha(a))$.

Obviously, $X \in \langle T \rangle$ if $X \in \text{add}(T)$. So we only need to consider the case that $X \notin \text{add}(T)$. It follows that

$$0 \neq \mathcal{C}(T, X[1]) = \mathcal{D}(T, X(\vec{\omega})) \oplus \mathcal{D}(T, X[1]).$$

If $\mathcal{D}(T, X[1]) \neq 0$, then $X \in \langle T \rangle$. If else, we get

$$\mathcal{D}(T, X(\vec{\omega})) \neq 0.$$

Let $\theta : T_X(-\vec{\omega}) \rightarrow X$ be a right $\text{add}(T(-\vec{\omega}))$ -approximation of X and

$$(3.4) \quad T_X(-\vec{\omega}) \xrightarrow{\theta} X \rightarrow Y \rightarrow T_X(-\vec{\omega})[1]$$

be the induced triangle in \mathcal{D} . By Lemma 3.2, $T_X(-\vec{\omega}) \in \langle T \rangle$. Hence we only need to show that $Y \in \langle T \rangle$. In fact, by similar arguments in Lemma 3.2, we obtain

$$(3.5) \quad \mathcal{D}(T(-\vec{\omega}), Y) = 0.$$

Since $\mu(X) \in (a, \alpha(a)]$ and $\mu(T_X(-\vec{\omega})[1]) \in (\alpha(a), \alpha^2(a)]$, we know that for each indecomposable direct summand Y_i of Y ,

$$\mu Y_i \in (a, \alpha^2(a)].$$

Case 1: $\mu Y_i \in (a, \alpha(a)]$. Then from (3.5), $\mathcal{C}(T, Y_i[1]) = \mathcal{D}(T, Y_i[1])$. Applying $\mathcal{D}(T[-1], -)$ to the triangle (3.4), we get $\mathcal{D}(T[-1], Y) = 0$, which implies $\mathcal{C}(T, Y_i[1]) = 0$. Hence $Y_i \in \text{add}(T)$.

Case 2: $\mu Y_i \in (\alpha(a), \alpha^2(a)]$. Then from (3.5), $\mathcal{C}(T, Y_i(\vec{\omega})) = \mathcal{D}(T, Y_i(2\vec{\omega})[-1])$. Applying $\mathcal{D}(T(-2\vec{\omega})[1], -)$ to the triangle (3.4), we get $\mathcal{D}(T(-2\vec{\omega})[1], Y) = 0$, which implies $\mathcal{C}(T, Y_i(\vec{\omega})) = 0$. Thus $Y_i(\vec{\omega})[-1] \in \text{add}(T)$, i.e., there exists some $T' \in \text{add}(T)$ such that $Y_i = T'(-\vec{\omega})[1]$. Hence by Lemma 3.2, $Y_i \in \langle T \rangle$, as claimed.

Conversely, let T be a tilting object in \mathcal{D} . By Lemma 3.1, we only need to prove that $\mathcal{C}(T, X[1]) \neq 0$ with an indecomposable X not occurring as a direct summand of T . Since there exists some $n \in \mathbb{Z}$ such that $\mu G^i X \in (a, \alpha(a)]$, without loss of generality, we assume $\mu X \in (a, \alpha(a)]$. Then by Lemma 2.3

$$\mathcal{C}(T, X[1]) = \mathcal{D}(T, X(\vec{\omega})) \oplus \mathcal{D}(T, X[1]).$$

It is sufficient to show that $\mathcal{D}(T, X[1]) = 0$ implies $\mathcal{D}(T, X(\vec{\omega})) \neq 0$. Note that $\langle T \rangle = \mathcal{D}$ and $\mathcal{D}(T, X[n]) = 0$ for $n \neq 0, 1$, the assumption $\mathcal{D}(T, X[1]) = 0$ implies that $\mathcal{D}(T, X) \neq 0$. Let $\theta : T_X \rightarrow X$ be a right $\text{add}(T)$ -approximation of X and

$$(3.6) \quad T_X \xrightarrow{\theta} X \xrightarrow{\beta} Y \rightarrow T_X[1]$$

be the induced triangle in \mathcal{D} . As in Lemma 3.2, it is easy to know that

$$(3.7) \quad \mathcal{D}(T, Y) = 0.$$

Moreover, by applying $\mathcal{D}(T[n], -)$ to the triangle (3.6), we get

$$\mathcal{D}(T[n], Y) = 0 \text{ for any } n \neq 0, 1.$$

Then combining with (3.7), we have $\mathcal{D}(T[1], Y_i) \neq 0$ for each indecomposable direct summand Y_i of Y . Hence $\mu Y_i \in (\alpha(a), \alpha^2(a)]$. Let $\delta : T_Y[1] \rightarrow Y$ be the minimal right $\text{add}(T[1])$ -approximation of Y and

$$(3.8) \quad T_Y[1] \xrightarrow{\delta} Y \rightarrow Z \rightarrow T_Y[2]$$

be the induced triangle in \mathcal{D} . Applying $\mathcal{D}(T[1], -)$ to the triangle (3.8), we have

$$\mathcal{D}(T[1], Z) = 0.$$

Moreover, it is easy to check that

$$\mathcal{D}(T[n], Z) = 0 \text{ for any } n \neq 1, 2.$$

Again since T is tilting, for each indecomposable direct summand Z_j of Z , we have

$$\mathcal{D}(T[2], Z_j) \neq 0.$$

Thus

$$\mu(Z_j) > \alpha^2(a).$$

By Lemma 2.3, we have $\mathcal{D}(X, Z) = 0$. Hence there exists a morphism $\gamma : X \rightarrow T_Y[1]$, such that $\beta = \delta\gamma \neq 0$.

$$\begin{array}{ccccc}
 & & T_Y[1] & & \\
 & \nearrow \gamma & \downarrow \delta & & \\
 T_X & \xrightarrow{\theta} & X & \xrightarrow{\beta} & Y & \xrightarrow{\quad} & T_X[1] \\
 & & & & \downarrow & & \\
 & & & & Z & & \\
 & & & & \downarrow & & \\
 & & & & T_Y[2] & &
 \end{array}$$

Then by Lemma 2.4,

$$\mathcal{D}(T_Y, X(\vec{\omega})) = D\mathcal{D}(X, T_Y[1]) \neq 0.$$

This finishes the proof. \square

4. TILTING OBJECTS FOR WEIGHT TYPE $(2, 2, 2, 2; \lambda)$

This section dedicates to investigate the tilting objects in the stable category of vector bundles on a weighted projective line of weight type $(2, 2, 2, 2; \lambda)$. Throughout \mathcal{D} is the stable category $\underline{\text{vect}}\mathbb{X}$ of vector bundles and \mathcal{C} is the attached cluster category.

4.1. Canonical tilting object. Kussin, Lenzing and Meltzer [16] showed the existence of a tilting object in $\underline{\text{vect}}\mathbb{X}$ for the weighted projective lines of three weights. For type $(2, 2, 2, 2; \lambda)$, Chen, Lin and Ruan [8] constructed a tilting object in $\underline{\text{vect}}\mathbb{X}$ whose endomorphism algebra is a canonical algebra of type $(2, 2, 2, 2)$.

Let E be the object defined uniquely up to isomorphism as the central term of the almost-split sequence

$$0 \rightarrow \mathcal{O}(\vec{\omega}) \rightarrow E \rightarrow \mathcal{O} \rightarrow 0.$$

Set

$$F = E(\vec{\omega} + \vec{c})[-1].$$

For each $i = 1, \dots, 4$, let E_i be the central term of the non-split exact sequence

$$0 \rightarrow \mathcal{O}(\vec{\omega}) \rightarrow E_i \rightarrow \mathcal{O}(\vec{x}_i) \rightarrow 0.$$

We have the following easy observation.

Lemma 4.1. *For each $i = 1, \dots, 4$,*

$$0 \rightarrow \mathcal{O}(\vec{\omega}) \rightarrow F \rightarrow E(\vec{\omega} + \vec{x}_i) \rightarrow 0$$

is an exact sequence in $\text{coh}\mathbb{X}$.

Proof. Let $P(E(\vec{\omega} + \vec{c}))$ be the projective cover of $E(\vec{\omega} + \vec{c})$. By [8],

$$P(E(\vec{\omega} + \vec{c})) = \mathcal{O}(\vec{c}) \oplus \left(\bigoplus_{i=1}^4 \mathcal{O}(\vec{\omega} + \vec{x}_i) \right).$$

We have the following exact sequence in $\text{coh}\mathbb{X}$

$$(4.1) \quad 0 \rightarrow F \rightarrow P(E(\vec{\omega} + \vec{c})) \rightarrow E(\vec{\omega} + \vec{c}) \rightarrow 0.$$

Applying $\text{Hom}(-, \mathcal{O}(\vec{x}_i))$ and $\text{Hom}(-, E(\vec{\omega} + \vec{x}_i))$ to (4.1), we have

$$(4.2) \quad \text{Hom}(F, \mathcal{O}(\vec{x}_i)) = 0 \quad \text{and} \quad \text{Hom}(F, E(\vec{\omega} + \vec{x}_i)) = k.$$

We prove that any non-zero morphism $\phi : F \rightarrow E(\vec{\omega} + \vec{x}_i)$ is surjective with kernel $\mathcal{O}(\vec{\omega})$. In fact, $\text{Im } \phi$ is of rank one or two because it is a subbundle of $E(\vec{\omega} + \vec{x}_i)$. If $\text{rk}(\text{Im } \phi) = 1$, then $\text{Im } \phi = \mathcal{O}(\vec{x}_i)$ follows, contradicting (4.2). Hence $\text{rk}(\text{Im } \phi) = 2$. The only possibility of $\text{Im } \phi$ is $E(\vec{\omega} + \vec{x}_i)$, that is, ϕ is surjective. It follows that

$$\text{rk}(\text{Ker } \phi) = 1 \text{ and } \det(\text{Ker } \phi) = \vec{\omega},$$

which ensure that $\text{Ker } \phi = \mathcal{O}(\vec{\omega})$. This finishes the proof. \square

The following is the main result in [8].

Lemma 4.2. [8] *The object*

$$T_{\text{can}} = E \oplus \left(\bigoplus_{i=1}^4 E_i \right) \oplus F$$

is a tilting object in \mathcal{D} and the endomorphism algebra $\text{End}(T_{\text{can}})$ is a canonical algebra of type $(2, 2, 2, 2)$.

In fact,

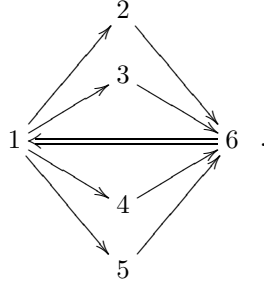
Lemma 4.3. *The object T_{can} is a cluster tilting object in \mathcal{C} .*

Proof. Since T_{can} is a direct sum of indecomposable vector bundles with slopes in the interval $(\alpha^{-1}(1), 1]$, by Theorem 3.3, we get what we want. \square

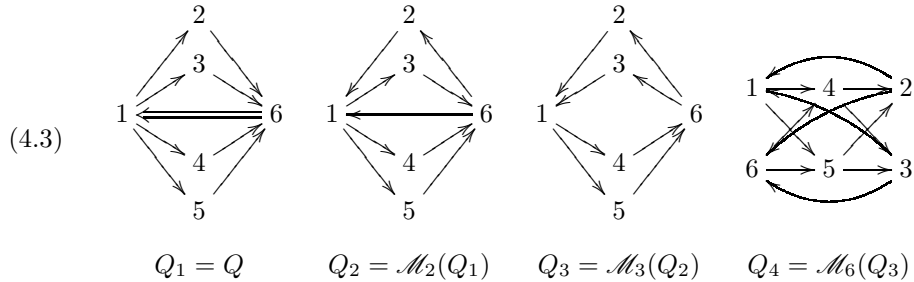
4.2. Quiver mutation and cluster tilting mutation. In the fundamental paper [9], Fomin and Zelevinsky introduced the notion of mutation of quivers as follows. Let Q be a finite quiver without loops and oriented cycles of length 2 (2-cycles for short). Let i be a vertex of Q . The mutation of the quiver Q at the vertex i is a quiver denoted by $\mathcal{M}_i(Q)$ and constructed from Q using the following rule:

- (M1) for any couple of arrows $j \rightarrow i \rightarrow k$, add an arrow $j \rightarrow k$;
- (M2) reverse the arrows incident with i ;
- (M3) remove a maximal collection of 2-cycles.

Example 4.4. *Let Q be the quiver*



It is easy to compute that the mutation class (the set of all quivers obtained from Q by iterated mutations) consists of the following 4 quivers up to isomorphism.



An advantage of cluster tilting theory over classical tilting theory is that from a cluster tilting object in a 2-Calabi-Yau triangulated category \mathcal{C} , it is possible to construct others by a recursive process resumed in the following.

Theorem 4.5 ([6, 12]). *Let \mathcal{C} be a Hom-finite 2-CY triangulated category with a cluster tilting object T . Let T_i be indecomposable and $T = T_0 \oplus T_i$. Then there exists a unique indecomposable T_i^* non-isomorphic to T_i such that $T_0 \oplus T_i^*$ is cluster tilting. Moreover T_i and T_i^* are linked by the existence of exchange triangles*

$$T_i \xrightarrow{u} B \xrightarrow{v} T_i^* \xrightarrow{w} T_i[1] \quad \text{and} \quad T_i^* \xrightarrow{u'} B' \xrightarrow{v'} T_i \xrightarrow{w'} T_i^*[1]$$

where u and u' are minimal left $\text{add } T_0$ -approximations and v and v' are minimal right $\text{add } T_0$ -approximations.

This recursive process of mutation of cluster tilting objects is closely related to the notion of mutation of quivers in the following sense.

Theorem 4.6 ([5]). *Let \mathcal{C} be a Hom-finite 2-CY triangulated category with a cluster tilting object T . Let T_i be an indecomposable direct summand of T , and denote by T' the cluster tilting object $\mathcal{M}_{T_i}(T)$. Denote by Q_T (resp. $Q_{T'}$) the quiver of the endomorphism algebra $\text{End}_{\mathcal{C}}(T)$ (resp. $\text{End}_{\mathcal{C}}(T')$). Assume that there are no loops and no 2-cycles at the vertex i of Q_T (resp. $Q_{T'}$) corresponding to the indecomposable T_i (resp. T_i^*). Then we have*

$$Q_{T'} = \mathcal{M}_i(Q_T),$$

where \mathcal{M}_i is the Fomin-Zelevinsky quiver mutation.

We shall make frequent use of the following results.

Lemma 4.7. *For each $i = 1, \dots, 4$, there is a triangle in \mathcal{C}*

$$E_i \xrightarrow{a_i} F \rightarrow E(\vec{\omega} + \vec{x}_i) \rightarrow E_i[1].$$

Proof. For any i , there is an almost split sequence in $\text{coh } \mathbb{X}$

$$(4.4) \quad 0 \rightarrow \mathcal{O}(\vec{x}_i) \xrightarrow{i} E(\vec{\omega} + \vec{x}_i) \rightarrow \mathcal{O}(\vec{\omega} + \vec{x}_i) \rightarrow 0.$$

Applying the functor $\text{Hom}(-, \mathcal{O}(\vec{\omega}))$ to (4.4), we get

$$\text{Ext}^1(E(\vec{\omega} + \vec{x}_i), \mathcal{O}(\vec{\omega})) = \text{Ext}^1(\mathcal{O}(\vec{x}_i), \mathcal{O}(\vec{\omega})).$$

By Lemma 4.1, there exists a commutative diagram induced by pullback of i and ϕ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(\vec{\omega}) & \longrightarrow & E_i & \longrightarrow & \mathcal{O}(\vec{x}_i) \longrightarrow 0 \\ & & \parallel & & \downarrow a_i & \circlearrowleft & \downarrow i \\ 0 & \longrightarrow & \mathcal{O}(\vec{\omega}) & \longrightarrow & F & \xrightarrow{\phi} & E(\vec{\omega} + \vec{x}_i) \longrightarrow 0. \end{array}$$

Moreover, we know that the right square is also a pushout. Hence $a_i : E_i \rightarrow F$ is injective and $\text{Coker}(a_i) = \text{Coker}(i) = \mathcal{O}(\vec{\omega} + \vec{x}_i)$. Then we obtain the following exact sequence:

$$(4.5) \quad 0 \rightarrow E_i \xrightarrow{a_i} F \rightarrow \mathcal{O}(\vec{\omega} + \vec{x}_i) \rightarrow 0.$$

Denote by $I(E_i)$ the injective hull of E_i . By [8],

$$I(E_i) = \mathcal{O}(\vec{x}_i) \oplus \left(\bigoplus_{j \neq i} \mathcal{O}(\vec{\omega} + \vec{x}_j) \right).$$

There is the following pushout

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_i & \longrightarrow & F & \longrightarrow & \mathcal{O}(\vec{\omega} + \vec{x}_i) \longrightarrow 0 \\ & & \downarrow & \circlearrowleft & \downarrow & & \parallel \\ 0 & \longrightarrow & I(E_i) & \longrightarrow & C & \longrightarrow & \mathcal{O}(\vec{\omega} + \vec{x}_i) \longrightarrow 0. \end{array}$$

Notice that for any $j \neq i$,

$$\mathrm{Ext}^1(\mathcal{O}(\vec{\omega} + \vec{x}_i), \mathcal{O}(\vec{\omega} + \vec{x}_j)) = 0.$$

Combining with (4.4), we get

$$C = E(\vec{\omega} + \vec{x}_i) \oplus \left(\bigoplus_{j \neq i} \mathcal{O}(\vec{\omega} + \vec{x}_j) \right).$$

Hence, there exists a triangle in \mathcal{D}

$$E_i \rightarrow F \rightarrow E(\vec{\omega} + \vec{x}_i) \rightarrow E_i[1].$$

Since $\pi : \mathcal{D} \rightarrow \mathcal{C}$ is a triangulated functor, we get what we want. \square

Lemma 4.8. *The following is a triangle in \mathcal{C}*

$$F[-1] \rightarrow E(\vec{x}_1 - \vec{x}_2) \rightarrow E_3 \oplus E_4 \rightarrow F.$$

Proof. From (4.5), we obtain the following commutative diagram induced by the pullback of a_3 and a_4

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{b_4} & E_4 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow b_3 & \circlearrowleft & \downarrow a_4 & & \downarrow b \\ 0 & \longrightarrow & E_3 & \xrightarrow{a_3} & F & \longrightarrow & \mathcal{O}(\vec{\omega} + \vec{x}_3) \longrightarrow 0, \end{array}$$

where the induced maps b_3, b_4 and b are all injective. Notice that

$$\mu E_4 = \frac{1}{2} \text{ and } \mu(\mathcal{O}(\vec{\omega} + \vec{x}_3)) = 1.$$

We get $B = 0$ or $B = \mathcal{O}(\vec{\omega} + \vec{x}_3)$. The fact $\mathrm{Hom}(E_4, E_3) = 0$ ensures that

$$B = \mathcal{O}(\vec{\omega} + \vec{x}_3).$$

It follows that the kernel A satisfying $\det A = \vec{x}_3 - \vec{x}_4$ and $\mathrm{rk} A = 1$. Hence $A = \mathcal{O}(\vec{x}_3 - \vec{x}_4)$, and the left square is also a pushout. Thus there is an exact sequence in $\mathrm{coh} \mathbb{X}$

$$0 \rightarrow \mathcal{O}(\vec{x}_3 - \vec{x}_4) \rightarrow E_3 \oplus E_4 \rightarrow F \rightarrow 0.$$

Denote by $P(F)$ the projective cover of F . By [8],

$$P(F) = \mathcal{O}(\vec{\omega})^2 \oplus \left(\bigoplus_{j \neq i} \mathcal{O}(\vec{x}_i - \vec{x}_j) \right).$$

Now consider the following pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(\vec{x}_3 - \vec{x}_4) & \longrightarrow & C & \longrightarrow & P(F) \longrightarrow 0 \\ & & \parallel & & \downarrow & \circlearrowleft & \downarrow \\ 0 & \longrightarrow & \mathcal{O}(\vec{x}_3 - \vec{x}_4) & \longrightarrow & E_3 \oplus E_4 & \longrightarrow & F \longrightarrow 0. \end{array}$$

It is easy to see that for any $\vec{x} \in \mathbb{L}$ with $\delta(\vec{x}) = 0$,

$$\mathrm{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x}_3 - \vec{x}_4)) \neq 0 \text{ if and only if } \vec{x} = \vec{x}_1 - \vec{x}_2.$$

Thus in \mathcal{D} ,

$$C = E(\vec{x}_1 - \vec{x}_2).$$

Similarly to the proof of Lemma 4.1, it is easy to know that there exists the following triangle in \mathcal{C}

$$F[-1] \rightarrow E(\vec{x}_1 - \vec{x}_2) \rightarrow E_3 \oplus E_4 \rightarrow F.$$

□

In [16], particular attention was given to the indecomposable vector bundles of rank two, which led to all major results there. The following question was raised by H.Lenzing.

Question 4.9. *Whether one can construct a tilting object in $\underline{\text{vect}}\mathbb{X}$ of weight type $(2, 2, 2, 2; \lambda)$, consisting of rank two bundles?*

The following is the main result in this subsection, which gives a positive answer to this question.

Theorem 4.10. *The object*

$$T_{\text{rk}} = E \oplus E(\vec{x}_1 - \vec{x}_2) \oplus \left(\bigoplus_{i=3}^4 E_i \right) \oplus \left(\bigoplus_{j=1}^2 E(\vec{\omega} + \vec{x}_j) \right)$$

is a tilting object in \mathcal{D} , with the quiver of the endomorphism algebra $\text{End}_{\mathcal{D}}(T_{\text{rk}})$ as the following

$$\Gamma_{\mathcal{D}} : \begin{array}{ccccc} E & \xrightarrow{\quad} & E_3 & \xrightarrow{\quad} & E(\vec{\omega} + \vec{x}_1) \\ & \searrow & \nearrow & \searrow & \nearrow \\ & & E_4 & \xrightarrow{\quad} & E(\vec{\omega} + \vec{x}_2) \\ E(\vec{x}_1 - \vec{x}_2) & \xrightarrow{\quad} & & & \end{array}$$

Proof. By Lemma 4.3,

$$T_{\text{can}} = E \oplus \left(\bigoplus_{i=1}^4 E_i \right) \oplus F = E_1 \oplus \overline{T}$$

is a cluster tilting object in \mathcal{C} . And the quiver of endomorphism algebra $\text{End}_{\mathcal{C}}(T_{\text{can}})$ has the form

$$Q_1 : \begin{array}{ccc} & E_1 & \\ & \nearrow & \searrow a_1 \\ & E_2 & \\ E & \xleftrightarrow{\quad} & F \\ & \searrow & \nearrow a_3 \\ & E_3 & \\ & \nearrow & \searrow a_4 \\ & E_4 & \end{array}$$

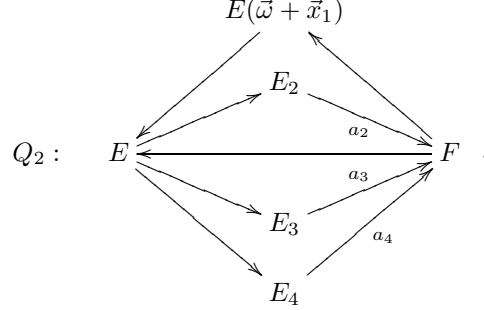
By Lemma 4.7, there exists a triangle in \mathcal{C}

$$E_1 \xrightarrow{a_1} F \rightarrow E(\vec{\omega} + \vec{x}_1) \rightarrow E_1[1],$$

where $a_1 : E_1 \rightarrow F$ is the minimal left add \overline{T} -approximation easily known from the above quiver. Then by Theorem 4.5,

$$T_{\star} = E \oplus \left(\bigoplus_{i=2}^4 E_i \right) \oplus F \oplus E(\vec{\omega} + \vec{x}_1)$$

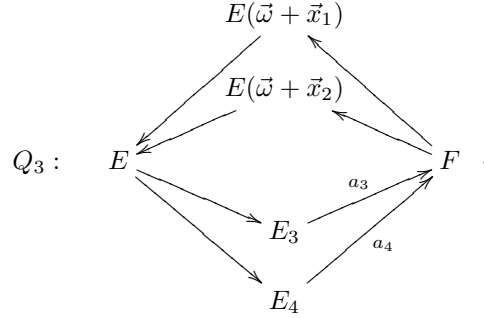
is a cluster tilting object in \mathcal{C} . By Theorem 4.6, we obtain the quiver of $\text{End}_{\mathcal{C}}(T_{\star})$ as follows



Write T_{\star} as $T_{\star} = E_2 \oplus \overline{T_{\star}}$. Similarly, we obtain that $a_2 : E_2 \rightarrow F$ is the minimal left $\text{add } \overline{T_{\star}}$ -approximation in \mathcal{C} and

$$T_{\star\star} = E \oplus \left(\bigoplus_{i=3}^4 E_i \right) \oplus F \oplus \left(\bigoplus_{j=1}^2 E(\vec{\omega} + \vec{x}_j) \right)$$

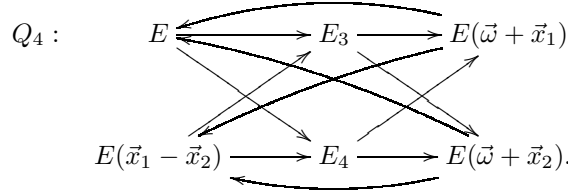
is a cluster tilting object, whose endomorphism algebra $\text{End}_{\mathcal{C}}(T_{\star\star})$ has the quiver



Write $T_{\star\star}$ as $F \oplus \overline{T_{\star\star}}$. By Lemma 4.8, there exists a triangle in \mathcal{C}

$$F[-1] \rightarrow E(\vec{x}_1 - \vec{x}_2) \rightarrow E_3 \oplus E_4 \xrightarrow{(a_3, a_4)} F,$$

where $(a_3, a_4) : E_3 \oplus E_4 \rightarrow F$ is the minimal right $\text{add } \overline{T_{\star\star}}$ -approximation easily known from the above quiver. Hence T_{rk} is a cluster tilting object in \mathcal{C} whose endomorphism algebra $\text{End}_{\mathcal{C}}(T_{\text{rk}})$ is given by



Notice that $T_{\text{rk}} \in \mathcal{D}(\alpha^{-1}(1), 1]$, applying Theorem 3.3 again completes the proof. \square

Remark 4.11. In the proof, we obtain four cluster tilting objects $T_{\text{can}}, T_{\star}, T_{\star\star}, T_{\text{rk}}$ in \mathcal{C} . As Example 4.4 indicated, the set of quivers of their endomorphism algebras is the mutation class of Q . Notice that the tilting graph of \mathcal{C} is connected (see for instance [2]), hence for any cluster tilting object T in \mathcal{C} , the quiver of the endomorphism algebra $\text{End}_{\mathcal{C}}(T)$ belongs to list (4.3).

4.3. Further relationship between tilting and cluster tilting objects. In this subsection, we investigate further relationship between the tilting objects in \mathcal{D} and the cluster tilting objects in \mathcal{C} for \mathbb{X} of weight type $(2,2,2,2;\lambda)$. Namely, under the canonical functor $\pi : \mathcal{D} \rightarrow \mathcal{C}$, tilting becomes cluster tilting. And on the other hand, we describe all the tilting objects corresponding to a given cluster tilting object.

For any $a \in \mathbb{Q}$, let $\Phi_a : \mathcal{D}(a, \alpha(a)] \rightarrow \text{coh } \mathbb{X}$ be the equivalence introduced in Lemma 2.5. The following lemma shows that Φ_a preserves the order of slopes.

Lemma 4.12. *For any indecomposable objects $F_1, F_2 \in \mathcal{D}(a, \alpha(a)]$,*

$$\mu F_1 < \mu F_2 \text{ if and only if } \mu(\Phi_a(F_1)) < \mu(\Phi_a(F_2)).$$

Consequently,

$$\mu F_1 = \mu F_2 \text{ if and only if } \mu(\Phi_a(F_1)) = \mu(\Phi_a(F_2)).$$

Proof. Notice that the rank of each tube in the Auslander-Reiten quiver of \mathcal{D} or $\text{coh } \mathbb{X}$ is less than or equal to two. We know that F_1 and F_2 are in the same tube if and only if

$$\mathcal{D}(F_i, \tau F_j \oplus F_j) \neq 0 \text{ for } i \neq j,$$

equivalently,

$$\text{Hom}(\Phi_a(F_i), \Phi_a(F_j) \oplus \tau(\Phi_a(F_j))) \neq 0 \text{ for } i \neq j,$$

that is, $\Phi_a(F_1)$ and $\Phi_a(F_2)$ are in the same tube.

For the necessity, firstly we claim that $\mu F_1 < \mu F_2$ implies $\mu(\Phi_a(F_1)) \leq \mu(\Phi_a(F_2))$. Otherwise, by Riemann-Roch Formula (2.2),

$$\begin{aligned} & \dim_k \text{Hom}(\Phi_a(F_2), \Phi_a(F_1) \oplus \tau(\Phi_a(F_1))) \\ &= \langle \Phi_a(F_2), \Phi_a(F_1) \oplus \tau(\Phi_a(F_1)) \rangle \\ &= \text{rk}(\Phi_a(F_1)) \text{rk}(\Phi_a(F_2)) (\mu(\Phi_a(F_1)) - \mu(\Phi_a(F_2))) > 0, \end{aligned}$$

Hence by Lemma 2.5, $\mathcal{D}(F_2, F_1 \oplus \tau F_1) \neq 0$. It follows that $\mu F_1 \geq \mu F_2$, which is a contradiction.

Secondly, we show that $\mu(\Phi_a(F_1)) = \mu(\Phi_a(F_2))$ implies $\mu F_1 = \mu F_2$. There are two cases to consider:

Case 1: $\Phi_a(F_1)$ and $\Phi_a(F_2)$ are in the same tube. We have done.

Case 2: $\Phi_a(F_1)$ and $\Phi_a(F_2)$ are in different tubes. Without loss of generality, we suppose that $\mu F_1 < \mu F_2$ for contradiction. By the structure of the Auslander-Reiten quiver of \mathcal{D} , there exists $F_3 \in \mathcal{D}$ such that

$$\mu F_1 < \mu F_3 < \mu F_2 \text{ and } \mathcal{D}(F_1, F_3) \neq 0.$$

It follows that

$$\mu(\Phi_a(F_1)) \leq \mu(\Phi_a(F_3)) \leq \mu(\Phi_a(F_2)),$$

which implies $\mu(\Phi_a(F_1)) = \mu(\Phi_a(F_3))$. Moreover, since

$$\text{Hom}(\Phi_a(F_1), \Phi_a(F_3)) = \mathcal{D}(F_1, F_3) \neq 0,$$

we obtain that $\Phi_a(F_1)$ and $\Phi_a(F_3)$ are in the same tube. It follows that F_1 and F_3 are in the same tube, which is a contradiction.

We now prove the other hand. It suffices to prove that

$$\mu F_1 = \mu F_2 \text{ implies } \mu(\Phi_a(F_1)) = \mu(\Phi_a(F_2)),$$

since we have shown above that

$$\mu F_1 > \mu F_2 \text{ implies } \mu(\Phi_a(F_1)) > \mu(\Phi_a(F_2)).$$

For contradiction, we assume $\mu(\Phi_a(F_1)) < \mu(\Phi_a(F_2))$ without loss of generality. Then

$$\mathcal{D}(F_1, F_2 \oplus \tau F_2) = \text{Hom}(\Phi_a(F_1), \Phi_a(F_2) \oplus \tau(\Phi_a(F_2))),$$

which is not zero by Riemann-Roch Formula (2.2). It follows that F_1 and F_2 , furthermore $\Phi_a(F_1)$ and $\Phi_a(F_2)$, are in the same tube, a contradiction to the assumption. \square

The following result is crucial.

Lemma 4.13. *Let F_1, F_2 be indecomposable objects in $\mathcal{D}(a, \alpha(a)]$. If $\mu F_1 < \mu F_2$, then $\mathcal{D}(F_1, F_2 \oplus \tau F_2) \neq 0$.*

Proof. By Lemma 2.5, Lemma 4.12 and Riemann-Roch Formula (2.2),

$$\begin{aligned} \dim_k \mathcal{D}(F_1, F_2 \oplus \tau F_2) &= \dim_k \text{Hom}(\Phi_a(F_1), \Phi_a(F_2) \oplus \tau(\Phi_a(F_2))) \\ &= \langle \Phi_a(F_1), \Phi_a(F_2) \oplus \tau(\Phi_a(F_2)) \rangle \\ &= \text{rk}(\Phi_a(F_1)) \text{rk}(\Phi_a(F_2)) (\mu(\Phi_a(F_2)) - \mu(\Phi_a(F_1))) > 0. \end{aligned}$$

This finishes the proof. \square

4.3.1. *Projecting a tilting object.* Recall from [8] that each tilting object in \mathcal{D} consists of just six indecomposable direct summands. Let $T = \bigoplus_{i=1}^6 T_i$ be a tilting object in \mathcal{D} . Without loss of generality, we assume $\mu T_i \leq \mu T_{i+1}, i = 1, \dots, 5$.

Lemma 4.14. *For all the indecomposable direct summands T_i , the slope μT_i is in the range $a \leq \mu T_i \leq \alpha(a)$ for some $a \in \mathbb{Q}$.*

Proof. In fact, we show that for any i ,

$$\mu T_i \leq \mu(T_1[1]).$$

Otherwise, there exists some i and $n \geq 2$ such that

$$\mu T_i \leq \mu(T_1[n]) < \mu(T_1[1]).$$

If $\mu T_i \neq \mu(T_1[n])$, by Lemma 4.13,

$$\mathcal{D}(T_i, T_1[n] \oplus \tau(T_1[n])) \neq 0.$$

But the fact that T is a tilting object in \mathcal{D} implies that

$$\mathcal{D}(T_i, T_1[n]) = 0,$$

and

$$\mathcal{D}(T_i, \tau(T_1[n])) = D\mathcal{D}(T_i[n-1], T_i) = 0.$$

These give a contradiction. Thus

$$(4.6) \quad \mu T_i = \mu(T_1[n]).$$

For those $j < i$ with $\mu T_j < \mu(T_1[1])$, we claim that

$$(4.7) \quad \mu T_j = \mu T_1.$$

In fact, since

$$\mu T_i = \mu(T_1[n]) > \mu(T_j[n-1]) \geq \mu(T_j[1]),$$

by the similar arguments above, there exists some $k_j \geq 2$ such that $\mu T_i = \mu(T_j[k_j])$. Comparing with (4.6), we get $\mu(T_j) = \mu(T_1[n - k_j])$. Then (4.7) follows from the assumption $\mu T_1 \leq \mu T_j < \mu(T_1[1])$. By means of (4.6) and (4.7), we have shown that the slope of each indecomposable direct summand of T has the form $\mu(T_1[m])$ for some integer m .

By [8] the exceptional objects in \mathcal{D} lie in the bottom of tubes of rank two. The indecomposable direct summands of T are orthogonal to each other if they have the same slope. Combining with Lemma 2.3, we obtain

$$\mathcal{D}(T_j, T_k) \neq 0 \text{ if and only if } T_k = \tau T_j[1].$$

It is not hard to see that the endomorphism algebra $\text{End}(T)$ is not connected, which is impossible. This finishes the proof. \square

The next lemma shows that we can obtain a series of tilting objects from T .

Lemma 4.15. *For $1 \leq i \leq 5$, the object*

$$(\bigoplus_{j=1}^i GT_j) \oplus (\bigoplus_{j=i+1}^6 T_j)$$

is tilting in \mathcal{D} .

Proof. We only prove the object

$$T' = GT_1 \oplus (\bigoplus_{i=2}^6 T_i)$$

is tilting in \mathcal{D} , the others are similar.

The following two equalities,

$$\mathcal{D}(GT_1, T_i[n]) = D\mathcal{D}(T_i[n-2], T_1) = 0 \text{ for any } n \in \mathbb{Z}$$

and

$$\mathcal{D}(T_i, GT_1[n]) = D\mathcal{D}(T_1[n], T_i) = 0 \text{ for any } n \neq 0,$$

imply that T' is extension-free. Moreover, we know that $T_1 \in \langle T' \rangle$ since

$$\mathcal{D}(GT_1, T_1[2]) = D\mathcal{D}(T_1, T_1) \neq 0.$$

Hence $\mathcal{D} = \langle T \rangle \subseteq \langle T' \rangle \subseteq \mathcal{D}$, which implies $\langle T' \rangle = \mathcal{D}$. \square

The following shows that each tilting object in \mathcal{D} pushes to a cluster tilting object in \mathcal{C} .

Theorem 4.16. *The image of T in \mathcal{C} is a cluster tilting object.*

Proof. Lemma 4.14 implies that

$$\mu T_6 \leq \mu(T_1[1]).$$

There are two cases to consider.

Case 1: $\mu T_6 < \mu(T_1[1])$. Then all the indecomposable direct summands are of slopes in the interval $(\mu(T_6[-1]), \mu(T_6))$. Hence $\pi(T)$ is a cluster tilting object in \mathcal{C} by Theorem 3.3.

Case 2: $\mu T_6 = \mu(T_1[1])$. Let i be the largest index satisfying $\mu T_1 = \mu T_i$. Lemma 4.15 implies that

$$T'' = (\bigoplus_{j=1}^i GT_j) \oplus (\bigoplus_{j=i+1}^6 T_j)$$

is tilting in \mathcal{D} . Clearly, the slope of each indecomposable direct summand is in the interval $(\mu T_1, \mu(T_1[1]))$. Then by Theorem 3.3, $\pi(T'')$ is a cluster tilting object in \mathcal{C} . Note that T and T'' have the same image in \mathcal{C} . We get what we want. \square

4.3.2. *Lifting a cluster tilting.* By Theorem 3.3, there is a one to one correspondence from the set of tilting objects in $\mathcal{D}(a, \alpha(a)]$ to the set of cluster tilting objects in \mathcal{C} . Each cluster tilting object in \mathcal{C} has the form $\pi(T)$, where $T = \bigoplus_{i=1}^6 T_i$ is tilting in \mathcal{D} . As before, we always assume $\mu T_i \leq \mu T_{i+1}$ for $1 \leq i \leq 5$. A *lifting* of $\pi(T)$ to \mathcal{D} is an object X in \mathcal{D} with $\pi(X) = \pi(T)$. Obviously, T is a lifting of $\pi(T)$, and any other lifting has the form

$$\bigoplus_{i=1}^6 G^{k_i} T_i, \text{ where } k_i \in \mathbb{Z}.$$

Theorem 4.17. *Let $T' = \bigoplus_{i=1}^6 G^{k_i} T_i$ be a lifting of $\pi(T)$. Then T' is tilting in \mathcal{D} if and only if $k_i \geq k_j \geq k_i - 1$ whence $\mu T_i < \mu T_j$.*

Proof. Assume T' is tilting in \mathcal{D} and $\mu T_i < \mu T_j$. Then Lemma 4.13 implies that

$$\mathcal{D}(T_i, \tau T_j \oplus T_j) \neq 0.$$

Notice that

$$\mathcal{D}(G^{k_i} T_i, G^{k_j} T_j[k_i - k_j]) = \mathcal{D}(T_i, \tau^{k_i - k_j} T_j)$$

and

$$\mathcal{D}(G^{k_j} T_j, G^{k_i} T_i[k_j - k_i + 1]) = \mathcal{D}(T_j, \tau^{k_j - k_i} T_i[1]) = D\mathcal{D}(T_i, \tau^{k_i - k_j + 1} T_j).$$

Hence T' is extension-free implies that either $k_i - k_j = 0$ or $k_j - k_i + 1 = 0$, that is, $k_i \geq k_j \geq k_i - 1$.

Conversely, assume $\mu T_i < \mu T_j$ implies $k_i \geq k_j \geq k_i - 1$. Arrange the indecomposables T_i with same slope by k_i to ensure

$$k_1 \geq k_2 \geq \dots \geq k_6 \geq k_1 - 1.$$

If $k_1 = k_2 = \dots = k_6$, then $T' = G^{k_6} T$ is a tilting object in \mathcal{D} . If else, there exists some $1 \leq l \leq 5$, such that

$$k_1 = \dots = k_l > k_{l+1} = \dots = k_6 = k_1 - 1.$$

So

$$T' = G^{k_6} (G(T_1 \oplus \dots \oplus T_l) \oplus (T_{l+1} \oplus \dots \oplus T_6)).$$

By Lemma 4.15, T' is a tilting object in \mathcal{D} . □

Corollary 4.18. *Let $T' = \bigoplus_{i=1}^6 G^{k_i} T_i$ be a lifting of $\pi(T)$. If T' is a tilting object in \mathcal{D} , then $\mu(G^{k_i} T_i) \in (a, \alpha(a)]$ for any i and some $a \in \mathbb{Q}$ if and only if $k_i = k_j$ whence $\mu T_i = \mu T_j$.*

Proof. Assume that all the slopes $\mu(G^{k_i} T_i)$ belong to $(a, \alpha(a)]$ and $\mu T_i = \mu T_j$. For contradiction, we assume $k_i > k_j$ without loss of generality. Then

$$\mu(G^{k_i} T_i) = \mu(T_i[k_i]) \geq \mu(T_j[k_j + 1]) = \alpha(\mu(T_j[k_j])) = \alpha(\mu(G^{k_j} T_j)),$$

which gives a contradiction.

On the contrary, by Theorem 4.17, the tilting object T' has the form

$$T' = G^{k_6} (G(T_1 \oplus \dots \oplus T_l) \oplus (T_{l+1} \oplus \dots \oplus T_6)) \text{ for some } l.$$

Since $\mu T_i = \mu T_j$ implies $k_i = k_j$, we have $\mu T_l < \mu T_{l+1}$. Hence for any $1 \leq i \leq 6$, we have

$$\mu(G^{k_i} T_i) \in [\mu(G^{k_6} T_{l+1}), \mu(G^{k_6+1} T_l)] \subseteq (\mu(G^{k_6} T_l), \mu(G^{k_6+1} T_l)).$$

Set $a = \mu(G^{k_6} T_l)$. We get what we want. □

4.4. Endomorphism algebras of tilting complexes and Meltzer's list. In this subsection, we will provide complete classifications of endomorphism algebras of tilting complexes in $D^b(\text{coh } \mathbb{X})$ and tilting sheaves in $\text{coh } \mathbb{X}$. As before, let $T = \bigoplus_{i=1}^6 T_i$ be a tilting object in \mathcal{D} where $T_i \in \mathcal{D}(a, \alpha(a)]$ for some $a \in \mathbb{Q}$. Let $\Gamma_{\mathcal{E}}$ be the quiver of the endomorphism algebra $\text{End}_{\mathcal{E}}(T)$ and $\Gamma_{\mathcal{D}}$ be that of $\text{End}_{\mathcal{D}}(T)$ (in fact $\text{End}_{\mathcal{D}(a, \alpha(a)]}(T)$).

Lemma 4.19. *For any $i \neq j$,*

- (1) $\mathcal{D}(T_i, T_j) \neq 0$ if and only if $\mu T_i < \mu T_j$;
- (2) $\mathcal{E}(T_i, T_j) = 0$ if and only if $\mu T_i = \mu T_j$.

Proof. (1) By [8], the indecomposable direct summands of T lie in the bottom of tubes of rank two, and they are orthogonal to each other if they have the same slope. Hence $\mathcal{D}(T_i, T_j) \neq 0$ implies $\mu T_i < \mu T_j$.

Conversely, by Lemma 4.13, $\mu T_i < \mu T_j$ implies that

$$\mathcal{D}(T_i, T_j \oplus \tau T_j) \neq 0.$$

But

$$\mathcal{D}(T_i, \tau T_j) = D\mathcal{D}(T_j, T_i[1]) = 0.$$

Thus $\mathcal{D}(T_i, T_j) \neq 0$.

(2) Let $\mathcal{E}(T_i, T_j) = 0$. For contradiction, we assume $\mu T_i < \mu T_j$ without loss of generality. Then $\mathcal{D}(T_i, T_j) \neq 0$ by (1). It follows that $\mathcal{E}(T_i, T_j) \neq 0$, a contradiction.

On the contrary, $\mu T_i = \mu T_j$ implies that (T_i, T_j) is an orthogonal pair [8]. Hence

$$\mathcal{E}(T_i, T_j) = \mathcal{D}(T_i, T_j) \oplus \mathcal{D}(T_i, GT_j) = \mathcal{D}(T_i, T_j) \oplus D\mathcal{D}(T_j, T_i) = 0.$$

□

Let $s_i : T_i \rightarrow T_{l,i}$ be the minimal left $\text{add}(T \setminus T_i)$ -approximation of T_i in \mathcal{E} .

Lemma 4.20. *If $\Gamma_{\mathcal{E}}$ has no 2-cycles and there exist $i \neq j$ satisfying*

- (i) *the approximations s_i and s_j have the same target T' ;*
- (ii) *for each indecomposable direct summand T_m of T' ,*

$$\dim \mathcal{E}(T_i, T_m) = \dim \mathcal{E}(T_j, T_m) = 1;$$

then $\mu T_i = \mu T_j$.

Proof. For contradiction, we assume $\mu T_i < \mu T_j$ without loss of generality. Then Lemma 4.19(1) implies $\mathcal{D}(T_i, T_j) \neq 0$. So there exists a path ρ from T_i to T_j in $\Gamma_{\mathcal{D}}$ and then in $\Gamma_{\mathcal{E}}$. By condition (i), the length of ρ is greater than one. Hence there exists at least one indecomposable direct summand T_m of T' , such that

$$\mu T_i < \mu T_m < \mu T_j.$$

Furthermore, we claim that for any indecomposable summand T_n of T' ,

$$(4.8) \quad \mu T_i < \mu T_n < \mu T_j.$$

In fact, if $\mu T_n \geq \mu T_j$ for some n , according to condition (ii), we get

$$\dim \mathcal{D}(T_i, T_n) = \dim \mathcal{D}(T_j, T_n) = 1.$$

Then by (i), the composition $T_i \rightarrow T_j \rightarrow T_n$ vanishes, which induces an arrow from T_n to T_i in $\Gamma_{\mathcal{E}}$ since $\mathcal{E}(T, T)$ can be explained as a trivial-extension and then a relation-extension algebra of $\mathcal{D}(T, T)$, cf. [21, 1]. Hence a 2-cycle between T_i and T_n appears in $\Gamma_{\mathcal{E}}$, a contradiction. If $\mu T_n \leq \mu T_i$ for some n , then $\mathcal{D}(T_n, T_m) \neq 0$ by Lemma 4.19(1). Moreover, according to condition (ii),

$$\dim \mathcal{D}(G^{-1}T_j, T_n) = \dim \mathcal{D}(G^{-1}T_j, T_m) = 1.$$

Similar arguments show that a 2-cycle between T_j and T_m appears in $\Gamma_{\mathcal{C}}$, which is a contradiction. Thus the claim (4.8) holds. It follows that

$$\mathcal{C}(T_j, T') = \mathcal{D}(T_j, GT').$$

Hence the approximation $s_j : T_j \rightarrow T'$ in \mathcal{C} lifts to a triangle in \mathcal{D}

$$\varepsilon : T_j \xrightarrow{s_j} GT' \rightarrow T_j^* \rightarrow T_j[1].$$

Applying $\mathcal{D}(T_i, -)$ to ε , we obtain

$$(4.9) \quad \mathcal{D}(T_i, T_j^*[-1]) \neq 0.$$

But ε induces a triangle in \mathcal{C} :

$$\bar{\varepsilon} : T_j \xrightarrow{s_j} T_l \rightarrow T_j^* \rightarrow T_j[1].$$

By Theorem 4.5, T_j^* is a complement of the almost complete cluster tilting object $T \setminus T_j$. Note that $[2] = G^2$ in \mathcal{D} . Therefore

$$\mathcal{C}(T_i, T_j^*[-1]) = \mathcal{C}(T_i, T_j^*[1]) = 0,$$

which gives a contradiction to (4.9). This finishes the proof. \square

Now we can give a classification of all the endomorphism algebras of tilting complexes in $D^b(\text{coh } \mathbb{X})$. For a totally different approach, see [18, Theorem 10.4.1].

Theorem 4.21. *Let Σ be a finite dimensional k -algebra. Then Σ is an endomorphism algebra of a tilting complex in $D^b(\text{coh } \mathbb{X})$ if and only if Σ belongs to the following list.*

List 1 Endomorphism algebras of tilting complexes of type (2, 2, 2, 2)

algebra	quiver	relations
B_{11}		$b_3 a_3 = b_2 a_2 - b_1 a_1$ $b_4 a_4 = b_2 a_2 - \lambda b_1 a_1$
B_{12}		$c_1 b_1 = 0$ $c_2 b_2 = 0$ $(c_2 - c_1) b_3 = 0$ $(c_2 - \lambda c_1) b_4 = 0$
A_{13}		$a_1 c_1 = 0$ $c_2 b_2 = 0$ $(c_2 - c_1) b_3 = 0$ $(c_2 - \lambda c_1) b_4 = 0$

$$\begin{array}{ccc}
 & 3 & \\
 & \searrow b_3 & \\
 A_{14} & & 5 \xrightleftharpoons[c_2]{c_1} 0' \\
 & \nearrow b_4 & \\
 & 4 &
 \end{array}
 \begin{array}{l}
 a_1 \nearrow 1' \\
 a_2 \searrow 2'
 \end{array}
 \begin{array}{l}
 a_1 c_1 = 0 \\
 a_2 c_2 = 0 \\
 (c_2 - c_1) b_3 = 0 \\
 (c_2 - \lambda c_1) b_4 = 0
 \end{array}$$

$$\begin{array}{ccc}
 & & 1' \\
 & & \nearrow a_1 \\
 A_{15} & 4 \xrightarrow{b_4} 5 \xrightleftharpoons[c_2]{c_1} 0' & \xrightarrow{a_2} 2' \\
 & & \searrow a_3 \\
 & & 3'
 \end{array}
 \begin{array}{l}
 c_1 b_4 = 0 \\
 a_1 c_2 = 0 \\
 a_2 (c_2 - c_1) = 0 \\
 a_3 (c_2 - \lambda c_1) = 0
 \end{array}$$

$$\begin{array}{ccc}
 & & 1' \\
 & & \nearrow a_1 \\
 B_{13} & 5 \xrightleftharpoons[c_2]{c_1} 0' & \xrightarrow{a_2} 2' \\
 & & \searrow a_3 \\
 & & 3' \\
 & & \searrow a_4 \\
 & & 4'
 \end{array}
 \begin{array}{l}
 a_1 c_1 = 0 \\
 a_2 c_2 = 0 \\
 a_3 (c_2 - c_1) = 0 \\
 a_4 (c_2 - \lambda c_1) = 0
 \end{array}$$

$$\begin{array}{ccc}
 & 2 & \\
 & \nearrow a_2 & \searrow b_2 \\
 B_{21} & 0 \xrightarrow{a_3} 3 \xrightarrow{b_3} 5 \xrightarrow{d_1} 1'' \\
 & \searrow a_4 & \nearrow b_4 \\
 & 4 &
 \end{array}
 \begin{array}{l}
 b_4 a_4 = b_3 a_3 - b_2 a_2 \\
 d_1 (b_3 a_3 - \lambda b_2 a_2) = 0
 \end{array}$$

$$\begin{array}{ccc}
 2 & \searrow b_2 & 1'' \\
 & & \nearrow d_1 \\
 B_{22} & 3 \xrightarrow{b_3} 5 \xrightarrow{c} 0' & \searrow e_1 \\
 & \nearrow b_4 & \\
 & 4 &
 \end{array}
 \begin{array}{l}
 c b_2 = 0 \\
 e_1 d_1 b_3 = 0 \\
 (e_1 d_1 - \frac{\lambda-1}{\lambda} c) b_4 = 0
 \end{array}$$

$$\begin{array}{ccc}
 3 & \searrow b_3 & 1'' \\
 & & \nearrow d_1 \\
 A_{23} & 5 \xrightarrow{c} 0' \xrightarrow{a_2} 2' \\
 & \nearrow b_4 & \\
 & 4 &
 \end{array}
 \begin{array}{l}
 c b_3 = 0 \\
 e_1 d_1 b_4 = 0 \\
 a_2 (e_1 d_1 - \frac{\lambda-1}{\lambda} c) = 0
 \end{array}$$

$$\begin{array}{ccc}
 & & 1'' \\
 & & \nearrow d_1 \\
 A_{24} & 4 \xrightarrow{b_4} 5 \xrightarrow{c} 0' & \xrightarrow{a_2} 2' \\
 & & \searrow a_3 \\
 & & 3'
 \end{array}
 \begin{array}{l}
 a_2 c = 0 \\
 a_3 e_1 d_1 = 0 \\
 (e_1 d_1 - \frac{\lambda-1}{\lambda} c) b_4 = 0
 \end{array}$$

$$B_{23} \quad \begin{array}{ccccc} & & 1'' & & 2' \\ & d_1 \nearrow & & e_1 \searrow & \\ 5 & \xrightarrow{c} & 0' & \xrightarrow{a_2} & 3' \\ & & & a_3 \searrow & \\ & & & & 4' \end{array} \quad \begin{aligned} a_2 c &= 0 \\ a_3 e_1 d_1 &= 0 \\ a_4 (e_1 d_1 - \frac{\lambda-1}{\lambda} c) &= 0 \end{aligned}$$

$$B_{24} \quad \begin{array}{ccccccc} & & & 2' & & & \\ & & a_2 \nearrow & & b_2 \searrow & & \\ 1'' & \xrightarrow{e_1} & 0' & \xrightarrow{a_3} & 3' & \xrightarrow{b_3} & 5' \\ & & a_4 \searrow & & b_4 \nearrow & & \\ & & & 4' & & & \end{array} \quad \begin{aligned} b_4 a_4 &= b_3 a_3 - b_2 a_2 \\ (b_3 a_3 - \lambda b_2 a_2) e_1 &= 0 \end{aligned}$$

$$B_{31} \quad \begin{array}{ccccccc} & & 3 & & 1'' & & \\ & a_3 \nearrow & & b_3 \searrow & d_1 \nearrow & & \\ 0 & & & & 5 & & \\ & a_4 \searrow & & b_4 \nearrow & d_2 \searrow & & \\ & & 4 & & & & 2'' \end{array} \quad \begin{aligned} d_1 (b_4 a_4 - b_3 a_3) &= 0 \\ d_2 (b_4 a_4 - \lambda b_3 a_3) &= 0 \end{aligned}$$

$$B_{32} \quad \begin{array}{ccccccc} & & 3 & & 1'' & & \\ & b_3 \searrow & & d_1 \nearrow & e_1 \searrow & & \\ & & 5 & & 0' & & \\ & b_4 \nearrow & & d_2 \searrow & e_2 \nearrow & & \\ 4 & & & & 2'' & & \end{array} \quad \begin{aligned} (e_2 d_2 - e_1 d_1) b_3 &= 0 \\ (e_2 d_2 - \lambda e_1 d_1) b_4 &= 0 \end{aligned}$$

$$A_{33} \quad \begin{array}{ccccccc} & & 1'' & & & & \\ & d_1 \nearrow & & e_1 \searrow & & & \\ 4 & \xrightarrow{b_4} & 5 & \xrightarrow{a_3} & 3' & & \\ & & d_2 \searrow & & e_2 \nearrow & & \\ & & & 2'' & & & \end{array} \quad \begin{aligned} a_3 (e_2 d_2 - e_1 d_1) &= 0 \\ (e_2 d_2 - \lambda e_1 d_1) b_4 &= 0 \end{aligned}$$

$$B_{41} \quad \begin{array}{ccccccc} 0 & \xrightarrow{a_3} & 3 & \xrightarrow{f_1} & 1'' & & \\ & a_4 \nearrow & & g_2 \searrow & & & \\ & h_3 \searrow & & g_1 \nearrow & & & \\ 5'' & \xrightarrow{h_4} & 4 & \xrightarrow{f_2} & 2' & & \end{array} \quad \begin{aligned} g_1 a_4 &= 0 \\ f_2 a_4 &= 0 \\ f_1 h_3 - g_1 h_4 &= 0 \\ f_2 h_4 - \lambda g_2 h_3 &= 0 \end{aligned}$$

$$A_{42} \quad \begin{array}{ccccccc} & & 3 & \xrightarrow{f_1} & 1'' & & \\ & a_3 \nearrow & & g_1 \searrow & & & \\ & a_4 \searrow & & g_2 \nearrow & & & \\ 0 & & & & 5' & & \\ & & 4 & \xrightarrow{f_2} & 2' & & \end{array} \quad \begin{aligned} g_1 a_4 &= 0 \\ f_2 a_4 &= 0 \\ k_1 f_1 - k_2 g_2 &= 0 \\ k_1 g_1 - \lambda k_2 f_2 &= 0 \end{aligned}$$

Proof. Assume $\Sigma = \text{End}_{D^b(\text{coh } \mathbb{X})}(T_c)$ for some tilting complex T_c in $D^b(\text{coh } \mathbb{X})$. Since $D^b(\text{coh } \mathbb{X}) = \mathcal{D}$ by [16], we can view T_c as a tilting object in \mathcal{D} and $\Sigma = \text{End}_{\mathcal{D}}(T_c)$. By Theorem 4.16, $\pi(T_c)$ is a cluster tilting object in \mathcal{C} . Hence the quiver Γ of the endomorphism algebra $\text{End}_{\mathcal{C}}(\pi(T_c))$ belongs to list (4.3) according to Remark 4.11. By Theorem 3.3, we suppose $\pi(T_c) = \pi(T)$ for some $T = \bigoplus_{i=1}^6 T_i$, where $T_i \in \mathcal{D}(a, \alpha(a))$ for each i and some $a \in \mathbb{Q}$. As before, we assume $\mu T_i \leq \mu T_{i+1}$ for $1 \leq i \leq 5$.

If $\Gamma = Q_1$, then by Lemma 4.20 and 4.19,

$$\mu T_1 < \mu T_2 = \mu T_3 = \mu T_4 = \mu T_5 < \mu T_6.$$

By Theorem 4.17, T_c has the form (under the equivalence G^{k_6})

$$\bigoplus_{i=1}^6 T_i \text{ or } GT_1 \oplus \left(\bigoplus_{i=2}^5 G^{k_i} T_i \right) \oplus T_6,$$

where $k_i = 0$ or 1 for $2 \leq i \leq 5$. For some choice of the representatives for the arrows, Σ is isomorphic to B_{11} , B_{12} , A_{13} , A_{14} , A_{15} or B_{13} in list 1.

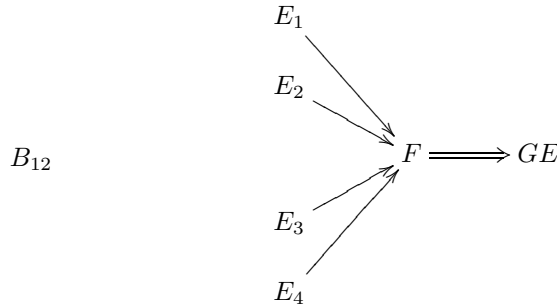
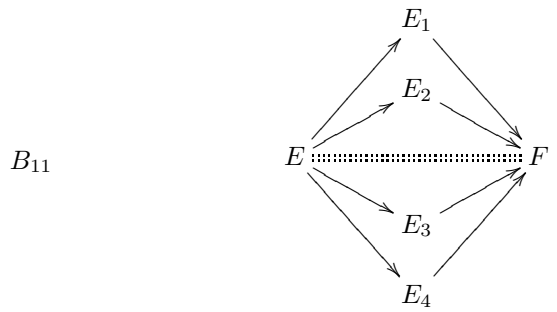
Similarly, one can prove that if $\Gamma = Q_2$, then Σ is isomorphic to B_{21} , B_{22} , A_{23} , A_{24} , B_{23} or B_{24} ; if $\Gamma = Q_3$, then Σ is isomorphic to B_{31} , B_{32} , or A_{33} ; if $\Gamma = Q_4$, then Σ is isomorphic to B_{41} or A_{42} in list 1.

Conversely, using the method applied to prove Theorem 6.2 in [8], combining Theorem 4.17 and Theorem 4.10, in each case in list 2 one proves that the listed objects do form a tilting object in \mathcal{D} having the corresponding algebras as endomorphism algebras.

List 2 Realization of the endomorphism algebras of tilting complexes of type (2, 2, 2, 2)

algebra

tilting complexes



$$A_{13} \quad \begin{array}{c} E_2 \\ E_3 \longrightarrow \\ E_4 \end{array} F \Longrightarrow GE \longrightarrow GE_1$$

$$A_{14} \quad \begin{array}{c} E_3 \\ E_4 \end{array} F \Longrightarrow GE \begin{array}{c} \nearrow GE_1 \\ \searrow GE_2 \end{array}$$

$$A_{15} \quad E_4 \longrightarrow F \Longrightarrow GE \begin{array}{c} \nearrow GE_1 \\ \longrightarrow GE_2 \\ \searrow GE_3 \end{array}$$

$$B_{13} \quad F \Longrightarrow GE \begin{array}{c} \nearrow GE_1 \\ \nearrow GE_2 \\ \searrow GE_3 \\ \searrow GE_4 \end{array}$$

$$B_{21} \quad \begin{array}{c} E \\ E \longrightarrow \\ E \end{array} \begin{array}{c} \nearrow E_2 \\ \longrightarrow E_3 \\ \searrow E_4 \end{array} F \longrightarrow E(\vec{\omega} + \vec{x}_1)$$

$$B_{22} \quad \begin{array}{c} E_2 \\ E_3 \longrightarrow \\ E_4 \end{array} F \begin{array}{c} \nearrow E(\vec{\omega} + \vec{x}_1) \\ \longrightarrow GE \end{array}$$

$$A_{23} \quad \begin{array}{c} E_3 \searrow \\ F \xrightarrow{E(\vec{\omega} + \vec{x}_1)} GE \longrightarrow GE_2 \\ E_4 \nearrow \end{array}$$

$$A_{24} \quad \begin{array}{c} E_4 \longrightarrow F \xrightarrow{E(\vec{\omega} + \vec{x}_1)} GE \begin{array}{l} \nearrow GE_2 \\ \searrow GE_3 \end{array} \end{array}$$

$$B_{23} \quad \begin{array}{c} F \xrightarrow{E(\vec{\omega} + \vec{x}_1)} GE \begin{array}{l} \nearrow GE_2 \\ \longrightarrow GE_3 \\ \searrow GE_4 \end{array} \end{array}$$

$$B_{24} \quad \begin{array}{c} E(\vec{\omega} + \vec{x}_1) \longrightarrow GE \begin{array}{l} \nearrow GE_2 \\ \longrightarrow GE_3 \\ \searrow GE_4 \end{array} \longrightarrow GF \end{array}$$

$$B_{31} \quad \begin{array}{c} E \begin{array}{l} \nearrow E_3 \\ \searrow E_4 \end{array} \begin{array}{l} \nearrow F \\ \searrow F \end{array} \begin{array}{l} \nearrow E(\vec{\omega} + \vec{x}_1) \\ \searrow E(\vec{\omega} + \vec{x}_2) \end{array} \end{array}$$

$$B_{32} \quad \begin{array}{c} E_3 \searrow \\ F \xrightarrow{E(\vec{\omega} + \vec{x}_1)} GE \\ E_4 \nearrow \end{array}$$

$$A_{33} \quad \begin{array}{c} E_4 \longrightarrow F \xrightarrow{E(\vec{\omega} + \vec{x}_1)} GE \longrightarrow GE_3 \\ \searrow E(\vec{\omega} + \vec{x}_2) \nearrow \end{array}$$

$$\begin{array}{c}
B_{41} \\
\begin{array}{ccccc}
E & \longrightarrow & E_3 & \longrightarrow & E(\vec{\omega} + \vec{x}_1) \\
& \searrow & \nearrow & \searrow & \nearrow \\
& & E_4 & \longrightarrow & E(\vec{\omega} + \vec{x}_2) \\
& \nearrow & \searrow & \nearrow & \searrow \\
E(\vec{x}_1 - \vec{x}_2) & \longrightarrow & E_4 & \longrightarrow & E(\vec{\omega} + \vec{x}_2)
\end{array} \\
\\
A_{42} \\
\begin{array}{ccccc}
& & E_3 & \longrightarrow & E(\vec{\omega} + \vec{x}_1) \\
& \nearrow & \searrow & \nearrow & \searrow \\
E(\vec{x}_1 - \vec{x}_2) & & & & GE \\
& \searrow & \nearrow & \searrow & \nearrow \\
& & E_3 & \longrightarrow & E(\vec{\omega} + \vec{x}_2)
\end{array}
\end{array}$$

□

4.5. Classification of Tilted algebras in the category of coherent sheaves.

This subsection presents classification result for the endomorphism algebras of tilting sheaves in $\text{coh } \mathbb{X}$. Let a be a rational number and Φ_a be the equivalence from the integral category $\mathcal{D}(a, \alpha(a))$ to $\text{coh } \mathbb{X}$.

Lemma 4.22. *Let T be an object in $\mathcal{D}(a, \alpha(a))$. Then T is tilting in \mathcal{D} if and only if $\Phi_a(T)$ is a tilting sheaf in $\text{coh } \mathbb{X}$.*

Proof. If $\Phi_a(T)$ is a tilting sheaf in $\text{coh } \mathbb{X}$, then T is clearly tilting in \mathcal{D} since $\mathcal{D} = D^b(\text{coh } \mathbb{X})$.

On the contrary, if T is a tilting object in \mathcal{D} , then

$$\text{Ext}_{\text{coh } \mathbb{X}}^1(\Phi_a(T), \Phi_a(T)) = \text{Ext}_{\mathcal{D}(a, \alpha(a))}^1(T, T) = \mathcal{D}(T, T[1]) = 0.$$

That is, $\Phi_a(T)$ is extension-free in $\text{coh } \mathbb{X}$. Moreover, for each object $X \in \text{coh } \mathbb{X}$ satisfying

$$\text{Ext}_{\text{coh } \mathbb{X}}^1(\Phi_a(T), X) = 0 = \text{Hom}_{\text{coh } \mathbb{X}}(\Phi_a(T), X),$$

we have

$$\mathcal{D}(T, \Phi_a^{-1}(X) \oplus \Phi_a^{-1}(X)[1]) = 0,$$

then it follows

$$\mathcal{D}(T, \bigoplus_{n \in \mathbb{Z}} \Phi_a^{-1}(X)[n]) = 0.$$

So the assumption that T is tilting in \mathcal{D} implies $\Phi_a^{-1}(X) = 0$, hence $X = 0$. This finishes the proof. □

Now we present detailed classification result.

Theorem 4.23. *Let Σ' be a finite dimensional k -algebra. Then Σ' is an endomorphism algebra of a tilting sheaf in $\text{coh } \mathbb{X}$ if and only if it is isomorphic to some B_{ij} in list 1.*

Proof. Since $\text{coh } \mathbb{X}$ is equivalent to the interval category $\mathcal{D}(a, \alpha(a))$, Σ' can be viewed as the endomorphism algebra of a tilting object in \mathcal{D} , with indecomposable direct summands are of slopes in the interval $(a, \alpha(a))$, which pushes to a cluster tilting object in \mathcal{C} by Theorem 3.3. We use the quivers of the endomorphism algebras of cluster tilting objects in \mathcal{C} to finish the classification. For a cluster tilting

object H in \mathcal{C} , we assume $H = \pi(\bigoplus_{i=1}^6 T_i)$ with $\mu T_i \leq \mu T_{i+1}$ for $1 \leq i \leq 5$ as before. If the quiver Γ of $\text{End}_{\mathcal{C}}(H)$ is Q_1 , by Lemma 4.20 and 4.19,

$$\mu T_1 < \mu T_2 = \mu T_3 = \mu T_4 = \mu T_5 < \mu T_6.$$

Then according to Corollary 4.18, a lifting of H in \mathcal{D} has one of the following forms (under the equivalence G^{k_6}):

$$\bigoplus_{i=1}^6 T_i, \quad GT_1 \oplus (\bigoplus_{i=2}^6 T_i), \quad (\bigoplus_{i=1}^5 GT_i) \oplus T_6.$$

For some choice of the representatives for the arrows, we obtain that the endomorphism algebras Σ' is isomorphic to B_{11} , B_{12} or B_{13} in list 1.

Similarly, one can prove that if $\Gamma = Q_2$, then Σ' is isomorphic to B_{21} , B_{22} , B_{23} or B_{24} ; if $\Gamma = Q_3$, then Σ' is isomorphic to B_{31} or B_{32} ; if $\Gamma = Q_4$, then Σ' is isomorphic to B_{41} .

On the other hand, each object corresponding to B_{ij} in list 2 gives a tilting sheaf that we need. \square

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